

Elementary Quantum Mechanics in a Space-Time Lattice

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Abstract

Studies of quantum fields and gravity suggest the existence of a minimal length, such as Planck length [1, 2]. It is natural to ask how existence of a minimal length may modify the results in elementary quantum mechanics (QM) problems familiar to us [3]. In this paper we address a familiar problem from elementary non-relativistic quantum mechanics, called “particle in a box”, where the usual continuum (1+1)-space-time is supplanted by a space-time lattice. Our lattice consists of a grid of $\lambda_0 \times \tau_0$ rectangles, where λ_0 , the lattice parameter, is a fundamental length (say Planck length) and, we take τ_0 to be equal to λ_0/c . The corresponding Schrodinger equation becomes a difference equation, the solution of which yields the q -eigenfunctions and q -eigenvalues of the energy operator as a function of λ_0 . The q -eigenfunctions form an ortho-normal set and both q -eigenfunctions and q -eigenvalues reduce to continuum solutions as $\lambda_0 \rightarrow 0$. The corrections to eigenvalues because of assumed lattice is shown to be $O(\lambda_0^2)$. We then compute the uncertainties in position and momentum, $\Delta x, \Delta p$ for the box problem and study the consequent modification of Heisenberg uncertainty relation due to the assumption of space-time lattice, in contrast to modifications suggested by other investigations such as [1].

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1 Introduction

It is widely believed [4] that a minimum length, of the order of the Planck length, results from combining quantum mechanics and classical general relativity. The key ingredients used to reach this conclusion are the uncertainty principle of quantum mechanics (QM) together with general relativity [2]. In recent years, considerable effort has been expended in exploring implications of the existence of a fundamental length, such as Planck length, in non-relativistic QM [5, 6].

Sometime ago, a discrete version of Quantum Mechanics, which may be called Finite Quantum Mechanics was introduced by Weyl [7]. This has enabled many investigations into the subject of discretized or finite Quantum Mechanics. Floratos and Leontaris [1] have investigated the possible evidence of the Hilbert space of Finite Quantum Mechanics and in particular, investigated the consequences for the Heisenberg uncertainty relation. The analysis of Gravitation, and its connection with the fundamental length has been studied by Mead [8] who established that it is not possible to detect the position of a particle with an accuracy better than a fundamental length. Adler and Santiago [9] have studied the effect of gravitation on the quantum Heisenberg uncertainty relation from a phenomenological theory of gravitation as well as a rigorous theory of gravitation. Kempf et al [2] have discussed the Hilbert space representation of the minimal length uncertainty relation. Such quantum gravity consequences are left for the reader to study in detail and we should cite all the references in these investigations.

In this paper we assume the existence of a fundamental length, λ_0 , as given regardless of its origin. We assume that λ_0 is the smallest measurable length in space dimension and $\tau_0 = \lambda_0/c$ is the smallest measurable interval for the time dimension. This assumption enables us to model (1+1) dimensional space-time as a lattice (grid) of $\lambda_0 \times \tau_0$ rectangles. We then propose to compare the solutions of Schrodinger equation for an elementary QM problem under two contrasting assumptions regarding space-time namely:

- (1) the usual (1+1) dimensional continuum for space-time.
- (2) a (1+1) dimensional space-time lattice (grid) of $\lambda_0 \times \tau_0$ rectangles.

The lattice described above is our (rather simplistic, if not naive) model for ‘quantized’ space-time. Under this model, we are interested in the dependence of eigenvalues, eigenfunctions, their orthogonality and normalization, on the lattice parameter, λ_0 . We also investigate the λ_0 -dependence of average uncertainties in position and in momentum for the problem at hand. This enables us to check what bearing, if any, λ_0 has on the Heisenberg uncertainty relation in our model.

In elementary non-relativistic quantum mechanics, under the assumption of space-time continuum, starting from the Hamiltonian of a one-dimensional particle in a potential $V(x)$, the time-dependent Schrodinger equation for the wave function $\psi(x, t)$ can be written as

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x) \psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t} \quad (1)$$

With $\psi(x, t) = T(t)u(x)$, (1) splits into two ordinary differential equations in the usual manner.

$$\frac{dT(t)}{dt} + i\omega T(t) = 0, \quad (2)$$

$$\frac{d^2 u(x)}{dx^2} + k^2 u(x) = 0, \quad (3)$$

where $\omega = \frac{E}{\hbar}$ and $k^2 = \frac{2m[E - V(x)]}{\hbar^2}$.

The general solution of Schrodinger equation (1), can be obtained once $V(x)$ and the relevant boundary conditions are specified.

The rest of the paper is organized as follows. In Section 2, we summarize the solution of the well-known ‘particle in a box’ problem under the usual assumption of space-time continuum. This is one of the standard problems discussed in introductory quantum mechanics textbooks (for example, [3]). In addition to eigenvalues and eigenfunctions for the problem in the continuum case, we also include the expressions for uncertainties in position and momentum, Δx and Δp , respectively. This is done for subsequent comparison of these quantities with the corresponding results for solutions obtained under the assumption of space-time lattice (hereafter called just lattice). We derive the eigenfunctions and eigenvalues under the lattice assumption

in Section 3 (denoting them as q -eigenvalues and q -eigenfunctions). We show that as the lattice parameter $\lambda_0 \rightarrow 0$, the q -eigenvalues and q -eigenfunctions approach the corresponding continuum eigenfunctions and eigenvalues. The uncertainties Δx and Δp under the lattice assumption are computed in Section 4. Again, we observe that as we proceed to the limit $\lambda_0 \rightarrow 0$, all the results for Δx and Δp obtained under the lattice assumption converge to the corresponding continuum results. A summary of our results is given in Section 5 and some concluding remarks and possible future investigations are given in Section 6.

2 Particle in a box - Solution with Space-Time Continuum

In this section we state the problem of ‘particle in a box’ in a notation similar to that of [3] and summarize the standard solution. The size of the box is denoted by L . The potential function $V(x)$ is defined by:

$$V(x) = \begin{cases} \infty & x \leq 0 \\ 0 & 0 < x < L \\ \infty & x \geq L \end{cases}$$

The nominal domain of the function $u(x)$ is the set of all reals, \mathbf{R} , but the interesting part of the domain is the interval $I = \{x \in \mathbf{R} \mid 0 \leq x \leq L\}$. The function $u(x)$ must satisfy the boundary conditions:

$$u(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 0 & \text{if } x \geq L \end{cases} . \quad (4)$$

Eigenvalues and Eigenfunctions with Space-time Continuum:

The solution of (2) is given by:

$$T(t) = T_0 e^{i\omega t} . \quad (5)$$

Setting $k^2 = \frac{2mE}{\hbar^2}$, ($E > 0$), the time independent Schrodinger equation (3) simplifies to the equation

$$\frac{d^2 u(x)}{dx^2} + k^2 u(x) = 0, \quad 0 < x < L \quad (6)$$

The general solution of (6) in the interval $0 < x < L$ subject to boundary conditions (4) under the assumption of a space-time continuum is discussed in elementary quantum mechanics textbooks (for example [3]), and can be written as:

$$u_n(x) = B [e^{ikx} - e^{-ikx}] , \quad (7)$$

$$i.e., u_n(x) = 2iB \sin(kx) \quad (8)$$

where eigenvalues for k are given by $k = \frac{\pi n}{L}$, $n = 1, 2, 3, \dots$

Substituting for k and setting $A = 2iB$, we can write the eigenfunctions in the usual form

$$u_n(x) = A \sin\left(\frac{n\pi x}{L}\right) \quad (9)$$

and exhibit the energy eigenvalues as:

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \quad n = 1, 2, 3, \dots \quad (10)$$

For $n \neq n'$, the eigenfunctions $u_n(x)$ and $u_{n'}(x)$ are orthogonal and the normalization constant A is determined to be

$$|A| = \sqrt{\frac{2}{L}}. \quad (11)$$

The uncertainties Δx and Δp can also be easily computed. We have:

$$\Delta x = \sqrt{|\langle x^2 \rangle - \langle x \rangle^2|} = \sqrt{\frac{n^2 \pi^2 - 6}{3}} \cdot \frac{L}{2n\pi} \quad (12)$$

$$\Delta p = \sqrt{|\langle p^2 \rangle - \langle p \rangle^2|} = \frac{n\pi\hbar}{L} \quad (13)$$

$$\Delta x \Delta p = \sqrt{\frac{n^2 \pi^2 - 6}{3}} \cdot (\hbar/2) \geq \frac{\hbar}{2} \quad (\text{for all } n \geq 1). \quad (14)$$

To no one's surprise, (14) verifies the validity of the Heisenberg uncertainty relation for the particle confined in a box.

3 Particle in a box - Solution with Space-Time Lattice

Next we proceed to obtain the solution of (6), with boundary conditions (4) under the assumption of quantized space-time.

We use the word “quantized” as synonymous with discreteized. Thus our quantized x -domain would simply be the discreteized real line given by $R_{\lambda_0} = \{j\lambda_0 \mid j \in \mathbf{Z}\}$, where λ_0 is the lattice parameter introduced earlier. (The lattice parameter λ_0 could be a “fundamental” length, such as the Planck length [10, 11, 9]) and j is an arbitrary integer. For simplicity, we express the quantized time domain in terms of $\tau_0 = \lambda_0/c$, c being the velocity of light. Thus $R_{\tau_0} = \{j_t\tau_0 \mid j_t \in \mathbf{Z}\}$. Note that for any integer j , the quantity $j\lambda_0$ corresponds to a particular point on the real line \mathbf{R} . Similarly $j_t\tau_0$ gives us a specific value for time t for any integer j_t . The continuum limit would obtain if $j \rightarrow \infty$. and $\lambda_0 \rightarrow 0$ then $j\lambda_0 \rightarrow x$, where x is a real number, $0 \leq x \leq L$. Similarly for j_t and $j_t\tau_0$.

Eigenvalues and Eigenfunctions with Space-time Lattice:

The time-dependence of functions $T(t)$ in the space-time setting can be presented as a simple (central¹) difference equation, (we use superscript q to denote results under our space-time lattice model) thus:

$$\frac{T^q((j_t + 1)\tau_0) - T^q((j_t - 1)\tau_0)}{2\tau_0} + i\omega T^q(j_t\tau_0) = 0 \quad (15)$$

or

$$T^q((j_t + 1)\tau_0) - T^q((j_t - 1)\tau_0) + 2i\tau_0\omega T^q(j_t\tau_0) = 0 \quad (16)$$

The solution of the above difference equation can be readily seen to be

$$T^q(j_t\tau_0) = T_0^q e^{ij_t\theta}, \quad \theta = \arctan \left(\frac{-w\tau_0}{\sqrt{1 - w^2\tau_0^2}} \right) \quad (17)$$

where we have set $T_0^q = T^q(0)$, the value of the function at time zero, in agreement with (5). We note that for small λ_0 (and therefore small τ_0 , the above solution reduces to:

$$T^q(j_t\tau_0) = T_0^q e^{-iwj_t\tau_0} \quad (18)$$

¹See for example [12]. It turns out the central difference representation for the differential operators leads to a hermitian representation for all relevant physical quantities.

We will not have much occasion to discuss the time dependence of eigenfunctions in this paper. We assume that the length L of the box is an integer multiple of λ_0 . Thus

$$L = J_0 \lambda_0 \quad (19)$$

where J_0 is a positive integer. For the quantized real line we can restate the boundary conditions (4) as

$$u^q(j\lambda_0) = \begin{cases} 0 & \text{if } j \leq 0 \\ 0 & \text{if } j \geq J_0, \end{cases} \quad (20)$$

where $j \in \mathbf{Z}$.

Next, we replace the ‘second derivative’ on the left hand side of (6) by the ‘second (central) difference’ with the prescription (see, for example, [12]):

$$\frac{d^2 u(x)}{dx^2} \longrightarrow \frac{u^q((j+2)\lambda_0) - 2u^q(j\lambda_0) + u^q((j-2)\lambda_0)}{4\lambda_0^2}. \quad (21)$$

It is significant to point out that we are employing the “central difference”. The various differences are defined in page 7 of Hildebrand [12]. Carrying out the replacement in (6) and multiplying through by $(4\lambda_0^2)$, we have:

$$u^q((j+2)\lambda_0) - 2(1 - 2k^2\lambda_0^2)u^q(j\lambda_0) + u^q((j-2)\lambda_0) = 0 \quad (22)$$

where

$$k^2 = \frac{2mE}{\hbar^2} \quad (23)$$

The above equation is a difference equation of second order with constant coefficients and can be solved by standard methods. The main steps are outlined below. We use the ansatz $u^q(j\lambda_0) = a^j$. Then the equation (22) becomes

$$a^{j+2} - 2(1 - 2k^2\lambda_0^2)a^j + a^{j-2} = 0 \quad (24)$$

from which we infer that

$$a = e^{i\theta/2}, \quad (25)$$

where θ is given by

$$\tan \theta = \pm \frac{2k\sqrt{1 - k^2}}{1 - 2k^2}. \quad (26)$$

The general solution of (22) can thus be exhibited as:

$$u^q(j\lambda_0) = Ae^{ij\theta/2} + Be^{-ij\theta/2}, \quad (27)$$

where θ is given by (26). Imposing the boundary conditions (19), we get

$$A + B = 0 \quad (28)$$

$$Ae^{iJ_0\theta/2} + Be^{-iJ_0\theta/2} = 0 \quad (29)$$

which lead to the eigenvalue equation:

$$e^{iJ_0\theta} = 1 \quad (30)$$

with the solution

$$\theta = \frac{2n\pi\lambda_0}{J_0}, \quad \text{with } n = 1, 2, 3, \dots \quad (31)$$

The Eigenvalue Equation with Space-time Lattice:

We thus obtain the eigenvalue equation:

$$\frac{2k\lambda_0\sqrt{1-k^2\lambda_0^2}}{1-2k^2\lambda_0^2} = \tan \frac{2n\pi\lambda_0}{L}. \quad (32)$$

We can solve (32) to obtain a closed expression for k^2 and consequently E_n as shown below. For convenience, we define

$$y = k^2\lambda_0^2 \quad \text{and} \quad t = \tan \left(\frac{2n\pi\lambda_0}{L} \right). \quad (33)$$

The equation (32) then becomes:

$$\frac{2\sqrt{y}\sqrt{1-y}}{1-2y} = t \quad (34)$$

and we obtain

$$y = \frac{1+t^2-\sqrt{1+t^2}}{2(1+t^2)}. \quad (35)$$

Substituting for y and t in (35), and observing that

$$1+t^2 = 1 + \tan^2 \left(\frac{2n\pi\lambda_0}{L} \right) = \sec^2 \left(\frac{2n\pi\lambda_0}{L} \right)$$

gives us the following equation.

$$k^2 = \frac{1}{2\lambda_0^2} \left(1 - \cos \left(\frac{2n\pi\lambda_0}{L} \right) \right) \quad (36)$$

$$\Rightarrow E_n^q = \frac{\hbar^2}{4m\lambda_0^2} \left(1 - \cos \left(\frac{2n\pi\lambda_0}{L} \right) \right), \quad n = 1, 2, 3, \dots \quad (37)$$

Thus, as in the continuum case, the variable k is discretized by the requirement that the solution of (22) satisfy the boundary conditions (28, 29). Expanding the right hand side of (37) in powers of λ_0 , we obtain

$$E_n^q = \frac{\hbar^2}{4m} \left(\frac{2n^2\pi^2}{L^2} - \frac{2n^4\pi^4}{3L^4} \lambda_0^2 + O(\lambda_0^4) \right) \quad (38)$$

$$= \frac{\pi^2 \hbar^2 n^2}{2mL^2} - \frac{\pi^4 \hbar^2 n^4}{6mL^4} \lambda_0^2 + O(\lambda_0^4), \quad n = 1, 2, 3, \dots \quad (39)$$

The second term in the previous equation is the leading correction to E_n , due to the assumed fundamental length λ_0 , which turns out to be $O(\lambda_0^2)$. We note that our results for correction to eigenvalues of ‘particle in box’ problem are consistent with the corrections of hydrogen atom spectrum (also $O(\lambda_0^2)$) obtained by Maziashvili et al [5] (treating it as a quantum-gravitational effect) and by Akhoury et al [6] (using minimum-length deformed QM)².

In the limit as $\lambda_0 \rightarrow 0$, we have $E_n^q \rightarrow E_n$, (see (10)) a result which serves as a check on our calculation.

Since k^2 is nonnegative, $\sqrt{k^2} = \pm k$ is a real number. With k determined by (36), the eigenfunctions $u_n^q(j\lambda_0)$ can be written as

$$u_n^q(j\lambda_0) = A^q \sin(n\pi j\lambda_0/L), \quad n = 1, 2, 3, \dots \quad (40)$$

Normalization and Orthogonality of q -Eigenfunctions :

To determine the normalization factor, A^q , and to check the orthogonality of the eigenfunctions $u_n^q(j\lambda_0)$, we have to compute the following sum. This is somewhat tedious to do by hand but use of *Mathematica* (or some other similar mathematical software) makes the analysis quite simple. We obtain:

$$S_{nn'} = \sum_{j=0}^{J_0} u_{n'}^{q*}(j\lambda_0) u_n^q(j\lambda_0) \quad (41)$$

$$= |A^q|^2 \sum_{j=0}^{J_0} [\sin(n'\pi j\lambda_0/L) \sin(n\pi j\lambda_0/L)] \quad (42)$$

$$= |A^q|^2 \left(\frac{L}{2} \right) \delta_{nn'} \quad (43)$$

where $\delta_{nn'}$ is the usual Kronekar delta symbol. It is then clear that the q -eigenfunctions $u_n^q(j\lambda_0)$ are indeed mutually orthogonal for $n \neq n'$ and that

²See [5] for a discussion of sign difference between the results for corrections obtained by [5] and [6]

the normalization factor $|A^q|$ is given by

$$|A^q| = \sqrt{\frac{2}{L}}$$

which turns out to be the same as in the continuum case.

4 Computation of Uncertainties with Space-Time Lattice

In the present section we calculate the uncertainties Δx and Δp with q -eigenfunctions in space-time lattice as given in (40). As in the continuum case, we compute

$$\langle p \rangle_n, \langle p^2 \rangle_n, \langle x \rangle_n, \text{ and } \langle x^2 \rangle_n$$

and using these we obtain Δx and Δp as usual.

Computation of Δp :

Recall that the operator p in central derivative representation takes the form

$$pu_n^q(j\lambda) \rightarrow -i\hbar \nabla u_n^q(j\lambda_0) \quad (44)$$

$$\rightarrow -i\hbar \frac{u_n^q((j+1)\lambda_0) - u_n^q((j-1)\lambda_0)}{2\lambda_0} \quad (45)$$

Using (40), we obtain

$$\langle p \rangle_n = \sum_{j=0}^{J_0} u_n^{q*}(j\lambda_0) pu_n^q(j\lambda_0) \quad (46)$$

$$= \frac{|A^q|^2}{2\lambda_0} \left[\sum_{j=1}^{J_0} \sin(n\pi(j-1)\lambda_0/L) \sin(n\pi j\lambda_0/L) \right] \quad (47)$$

$$= \sum_{j=1}^{J_0} \sin(n\pi j\lambda_0/L) \sin(n\pi(j-1)\lambda_0/L) \quad (48)$$

where we added and subtracted some terms to the two sums in (48), which vanish because of boundary conditions (20). Thus we see that $\langle p \rangle_n = 0$.

We can compute $\langle p^2 \rangle_n$ by evaluating the sum directly. We can also get the answer more readily by observing that

$$\langle p^2 \rangle_n = \langle 2mE \rangle_n = \hbar^2 k^2 = \frac{\hbar^2}{2\lambda_0^2} [1 - \cos(\frac{2n\pi\lambda_0}{L})] \quad (49)$$

Thus

$$(\Delta p)^2 = \langle p^2 \rangle_n - \langle p \rangle_n^2 \quad (50)$$

$$= \frac{n^2 \pi^2 \hbar^2}{L^2} - \frac{n^4 \pi^4 \hbar^2}{3L^4} \lambda_0^2 + O(\lambda_0^4) \quad (51)$$

and therefore

$$\Delta p = \left(\frac{\hbar}{\lambda_0} \right) \sin\left(\frac{n\pi\lambda_0}{L}\right) \quad (52)$$

$$= \frac{\pi n \hbar}{L} - \frac{\pi^3 n^3 \hbar}{6L^3} \lambda_0^2 + O(\lambda_0^4) \quad (53)$$

Computation of $\langle x \rangle_n$:

Since

$$u_n^{q*}(j\lambda_0) x u_n^q(j\lambda) \rightarrow u_n^{q*}(j\lambda_0) j \lambda_0 u_n^q(j\lambda_0) \quad (54)$$

using (40), and recalling that

$$a = (n\pi\lambda_0/L). \quad (55)$$

we obtain

$$\langle x \rangle_n = \lambda_0^2 \sum_{j=0}^{J_0} j u_n^{q*}(j\lambda_0) u_n^q(j\lambda_0) \quad (56)$$

$$= |A^q|^2 \lambda_0^2 \sum_{j=0}^{J_0} j \sin^2(a j) \quad (57)$$

The sum $\sum_{j=0}^{J_0} j \sin^2(a j)$ can be evaluated by standard techniques and we obtain:

$$\sum_{j=0}^{J_0} j \sin^2(a j) = \frac{L^2}{4} \quad (58)$$

where we have used the definition $J_0 = \frac{L}{\lambda_0}$ to simplify the final sum. Substituting for $|A^q|^2$ in (57), we get:

$$\langle x \rangle_n = \frac{L}{2} \quad (59)$$

Computation of $\langle x^2 \rangle_n$:

Now since

$$u_n^{q*}(j\lambda_0)x^2u_n^q(j\lambda) \rightarrow u_n^{q*}(j\lambda_0)(j\lambda)^2u_n^q(j\lambda_0) \quad (60)$$

using (40), we obtain

$$\langle x^2 \rangle_n = \lambda_0^3 \sum_{j=0}^{J_0} j^2 u_n^{q*}(j\lambda_0) u_n^q(j\lambda_0) \quad (61)$$

$$= |A^q|^2 \lambda_0^3 \sum_{j=0}^{J_0} j^2 \sin^2(a_j) \quad (62)$$

The sum $\sum_{j=0}^{J_0} j^2 \sin^2(a_j)$ can be evaluated by standard techniques and we obtain:

$$\sum_{j=0}^{J_0} j^2 \sin^2(a_j) = \frac{L^3}{6} + \frac{L\lambda_0^2}{12} - \left(\frac{L\lambda_0^2}{4}\right) \csc^2\left(\frac{\lambda_0 n \pi}{L}\right) \quad (63)$$

$$= \left(\frac{L^3}{6} - \frac{L^3}{4n^2\pi^2}\right) - \left(\frac{n^2\pi^2}{60L}\right) \lambda_0^4 - O(\lambda_0^6) \quad (64)$$

Substituting for $|A^q|^2$, and rearranging, we obtain:

$$\langle x^2 \rangle_n = \frac{L^2}{3} + \frac{\lambda_0^2}{6} - \left(\frac{\lambda_0^2}{2}\right) \csc^2\left(\frac{\lambda_0 n \pi}{L}\right) \quad (65)$$

Using (59) and (65), we get:

$$(\Delta x)^2 = \langle x^2 \rangle_n - (\langle x \rangle_n)^2 \quad (66)$$

$$= \left(\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}\right) - \left(\frac{n^2\pi^2}{30L^2}\right) \lambda_0^4 + O(\lambda_0^6) \quad (67)$$

which leads to

$$\Delta x = \left(\frac{1}{2\sqrt{3}}\right) \sqrt{L^2 + 2\lambda_0^2 - 6\lambda_0^2 \csc^2\left(\frac{\lambda_0 n \pi}{L}\right)} \quad (68)$$

$$= \left(\sqrt{\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}}\right) - \left(\frac{\frac{n^2\pi^2}{60}}{\left(L^2 \sqrt{\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}}\right)}\right) \lambda_0^4 + O(\lambda_0^6) \quad (69)$$

Using equations (68) and (44), we can write:

$$\Delta x \Delta p = \left(\frac{\hbar}{2\sqrt{3}\lambda_0} \right) \left(\sqrt{L^2 + 2\lambda_0^2 - 6\lambda_0^2 \csc^2\left(\frac{\lambda_0 n \pi}{L}\right)} \right) \sin\left(\frac{n\pi\lambda_0}{L}\right) \quad (70)$$

$$= \left[\left(\sqrt{\frac{n^2\pi^2-6}{3}} \right) - \left(\frac{n^2\pi^2\sqrt{\frac{n^2\pi^2-6}{3}}}{6L^2} \right) \lambda_0^2 \right] \left(\frac{\hbar}{2} \right) + O(\lambda_0^4) \quad (71)$$

$$= \sqrt{\frac{n^2\pi^2-6}{3}} \left[1 - \left(\frac{n^2\pi^2}{6L^2} \right) \lambda_0^2 \right] \left(\frac{\hbar}{2} \right) + O(\lambda_0^4) \quad (72)$$

The first term in the square bracket in (71) gives us the continuum result while the second term is the correction because of the assumed space-time lattice and is of second order in the lattice parameter λ_0 .

5 Particle in a box: Summary

With our model of quantized space-time as described in the foregoing, we have obtained the q -eigenvalues and q -eigenfunctions for the problem of a particle in a box as a function of the lattice parameter λ_0 . We have shown the q -eigenfunctions form an ortho-normal set. We have also computed the uncertainties Δx and Δp for the problem. The modification of the Heisenberg uncertainty relation induced by the assumption of space-time lattice for the box problem is shown in (70) and is of second order in lattice parameter λ_0 as presented in (72).

By examining the limit process as $\lambda_0 \rightarrow 0$, we can see that the q -eigenvalues reduce to the ones obtained for the case of continuum domain. Same holds for the q -eigenfunctions as also for the expressions for $\Delta x, \Delta p$. This reflects the continuous nature of the results for eigenfunctions, eigenvalues and uncertainties obtained for the space-time lattice, as a function of the lattice parameter λ_0 .

The first order modification of product $\Delta x \Delta p$ induced by the assumed space-time lattice is of the form

$$\Delta x \Delta p = \alpha \left(1 + \beta \lambda_0^2 \right) \left(\frac{\hbar}{2} \right)$$

where the coefficients α and β for the ‘particle in box’ problem, β can be

read off from (72) as

$$\alpha = \sqrt{\frac{n^2\pi^2 - 6}{3}} \quad (73)$$

$$\beta = -\left(\frac{n^2\pi^2}{6L^2}\right) \quad (74)$$

The parameters α and β , of course, depend on the specific problem that we are investigating. However the form of correction of the product of uncertainties above resembles the one given in [1] as well as other papers exploring consequences of minimum length in uncertainty relation (see, for example, [5] and [6]). It is interesting to note that even a simplistic model for space-time lattice and a semi-classical approach to QM seems to reveal shadows of deeper theories.

6 Concluding Remarks

Motivated by the possible existence of a fundamental length, such as Planck length, as a physical reality, we pose the question: how may a fundamental length modify the results in elementary quantum mechanics problems familiar to us? The investigation of the implications of space-time lattice from many different perspectives continues to draw considerable attention in literature (See for example [1] and reference cited there). In this paper, we have explored the “particle in a box” problem in some detail with a simple model for space time lattice and obtained corrections induced by the existence of a space-time lattice to the results for eigenvectors eigenvalues, average uncertainties given under the space-time continuum. We have exhibited how the Heisenberg uncertainty relation in space-time continuum compares with the corresponding result in space-time lattice.

For the “particle in a box” problem with space-time lattice, we found that the corrections to eigenvalues etc are of second order in the lattice parameter λ_0 , and although the corrections are observable in principle, the rather small magnitude of Planck length, the lattice parameter $\lambda_0 \approx 10^{-35}$ renders it difficult if not impossible to measure at present. All the same, what one can hope is that such an investigation might give us a pointer, however faint, to a future correct theory which incorporates fundamental length, quantum mechanics as well as theory of relativity. To that end, it appears that it would be useful to investigate other elementary quantum

mechanics problems ³. For instance, it would be interesting to bring out the λ_0 -dependence of eigenvalue spectrum and eigenfunctions of a simple harmonic oscillator or hydrogen atom in our model and compare them with other approaches to consequences of fundamental length in QM (see for example [5] and [6]). Equally interesting would be to investigate if λ_0 -dependence of any physical quantity is measurable in any experiment in the foreseeable future.

³For a brief discussion of the free particle under the lattice assumption, see [13].

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Elementary Quantum Mechanics in a Space-Time Lattice

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Abstract

Studies of quantum fields and gravity suggest the existence of a minimal length, such as Planck length [1, 2]. It is natural to ask how the existence of a minimal length may modify the results in elementary quantum mechanics (QM) problems familiar to us [3]. In this paper we address a simple problem from elementary non-relativistic quantum mechanics, called “particle in a box”, where the usual continuum (1+1)-space-time is supplanted by a space-time lattice. Our lattice consists of a grid of $\lambda_0 \times \tau_0$ rectangles, where λ_0 , the lattice parameter, is a fundamental length (say Planck length) and, we take τ_0 to be equal to λ_0/c . The corresponding Schrodinger equation becomes a difference equation, the solution of which yields the q -eigenfunctions and q -eigenvalues of the energy operator as a function of λ_0 . The q -eigenfunctions form an orthonormal set and both q -eigenfunctions and q -eigenvalues reduce to continuum solutions as $\lambda_0 \rightarrow 0$. The corrections to eigenvalues because of the assumed lattice is shown to be $O(\lambda_0^2)$. We then compute the uncertainties in position and momentum, $\Delta x, \Delta p$ for the box problem and study the consequent modification of Heisenberg uncertainty relation due to the assumption of space-time lattice, in contrast to modifications suggested by other investigations such as [1].

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1 Introduction

It is widely believed [4] that a minimum length, of the order of the Planck length, results from combining quantum mechanics and classical general relativity. The key ingredients used to reach this conclusion are the uncertainty principle of quantum mechanics (QM) together with general relativity [2]. In recent years, considerable effort has been expended in exploring implications of the existence of a fundamental length, such as Planck length, in non-relativistic QM [5, 6].

Sometime ago, a discrete version of Quantum Mechanics, which may be called Finite Quantum Mechanics was introduced by Weyl [7]. This has been followed by many investigations into the subject of discretized or finite Quantum Mechanics. Floratos and Leontaris [1] have investigated the possible evidence of the Hilbert space of Finite Quantum Mechanics and in particular, investigated the consequences for the Heisenberg uncertainty relation. The analysis of Gravitation, and its connection with the fundamental length has been studied by Mead [8] who established that it is not possible to detect the position of a particle with an accuracy better than a fundamental length. Adler and Santiago [9] have studied the effect of gravitation on the quantum Heisenberg uncertainty relation from a phenomenological theory of gravitation as well as a rigorous theory of gravitation. Kempf et al [2] have discussed the Hilbert space representation of the minimal length uncertainty relation.

In this paper we assume the existence of a fundamental length, λ_0 , as given regardless of its origin. We assume that λ_0 is the smallest measurable length in space dimension and $\tau_0 = \lambda_0/c$ is the smallest measurable interval for the time dimension. This assumption enables us to model (1+1) dimensional space-time as a lattice (grid) of $\lambda_0 \times \tau_0$ rectangles. We then propose to compare the solutions of Schrodinger equation for an elementary QM problem under two contrasting assumptions regarding space-time namely:

- (1) the usual (1+1) dimensional continuum for space-time.
- (2) a (1+1) dimensional space-time lattice (grid) of $\lambda_0 \times \tau_0$ rectangles.

The lattice described above is our (rather simplistic, if not naive) model for ‘quantized’ space-time. Under this model, we are interested in the dependence of eigenvalues, eigenfunctions, their orthogonality and normalization, on the lattice parameter, λ_0 . We also investigate the λ_0 -dependence of average uncertainties in position and in momentum for the problem at hand. This enables us to check what bearing, if any, λ_0 has on the Heisenberg uncertainty relation in our model.

In elementary non-relativistic quantum mechanics, under the assumption of space-time continuum, starting from the Hamiltonian of a one-dimensional particle in a potential $V(x)$, the time-dependent Schrodinger equation for the wave function $\psi(x, t)$ can be written as

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t} \quad (1)$$

With $\psi(x, t) = T(t)u(x)$, (1) splits into two ordinary differential equations in the usual manner.

$$\frac{dT(t)}{dt} + i\omega T(t) = 0, \quad (2)$$

$$\frac{d^2u(x)}{dx^2} + k^2u(x) = 0, \quad (3)$$

where

$$\omega = \frac{E}{\hbar} \quad (4)$$

$$k^2 = \frac{2m[E - V(x)]}{\hbar^2}. \quad (5)$$

The general solution of Schrodinger equation (1), can be obtained once $V(x)$ and the relevant boundary conditions are specified.

The rest of the paper is organized as follows. In Section 2, we summarize the solution of the well-known ‘particle in a box’ problem under the usual assumption of space-time continuum. This is one of the standard problems discussed in introductory quantum mechanics textbooks (such as [3]). In addition to eigenvalues and eigenfunctions for the problem in the continuum case, we also include the expressions for uncertainties in position and momentum, Δx and Δp , respectively. This is done for subsequent comparison of these quantities with the corresponding results for solutions obtained under the assumption of space-time lattice (hereafter called just lattice). We derive the eigenfunctions and eigenvalues under the lattice assumption in Section 3 (denoting them as q -eigenvalues and q -eigenfunctions). We show that as the lattice parameter $\lambda_0 \rightarrow 0$, the q -eigenvalues and q -eigenfunctions approach the corresponding continuum case eigenfunctions and eigenvalues. The uncertainties Δx and Δp under the lattice assumption are computed in Section 4. Again, we observe that as we proceed to the limit $\lambda_0 \rightarrow 0$, all the results for Δx and Δp obtained under the lattice assumption converge to the corresponding continuum results. A summary of our results is given in Section 5 and some concluding remarks and possible future investigations are given in Section 6.

2 Particle in a box - Solution with Space-Time Continuum

In this section we state the problem of ‘particle in a box’ in a notation similar to that of [3] and summarize the standard solution. The size of the box is denoted by L . The potential function $V(x)$ is defined by:

$$V(x) = \begin{cases} \infty & x \leq 0 \\ 0 & 0 < x < L \\ \infty & x \geq L \end{cases}$$

The nominal domain of the function $u(x)$ is the set of all reals, \mathbf{R} , but the interesting part of the domain is the interval $I = \{x \in R \mid 0 \leq x \leq L\}$. The function $u(x)$ must satisfy the boundary conditions:

$$u(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 0 & \text{if } x \geq L \end{cases} . \quad (6)$$

Eigenvalues and Eigenfunctions with Space-time Continuum:

The solution of (2) is given by:

$$T(t) = T_0 e^{i\omega t} . \quad (7)$$

Setting $k^2 = \frac{2mE}{\hbar^2}$, ($E > 0$), the time independent Schrodinger equation (3) simplifies to the equation

$$\frac{d^2 u(x)}{dx^2} + k^2 u(x) = 0, \quad 0 < x < L \quad (8)$$

The general solution of (8) in the interval $0 < x < L$ subject to boundary conditions (6) under the assumption of a space-time continuum is discussed in elementary quantum mechanics textbooks (for example [3]), and can be written as:

$$u_n(x) = B \left[e^{ikx} - e^{-ikx} \right] , \quad (9)$$

$$i.e., u_n(x) = 2iB \sin(kx) \quad (10)$$

where eigenvalues for k are given by $k = \frac{\pi n}{L}$, $n = 1, 2, 3, \dots$

Substituting for k and setting $A = 2iB$, we can write the eigenfunctions in the usual form

$$u_n(x) = A \sin\left(\frac{n\pi x}{L}\right) \quad (11)$$

and exhibit the energy eigenvalues as:

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2}, \quad n = 1, 2, 3, \dots \quad (12)$$

For $n \neq n'$, the eigenfunctions $u_n(x)$ and $u_{n'}(x)$ are orthogonal and the normalization constant A is determined to be

$$|A| = \sqrt{\frac{2}{L}} . \quad (13)$$

The uncertainties Δx and Δp can also be easily computed. We have:

$$\Delta x = \sqrt{|\langle x^2 \rangle - \langle x \rangle^2|} = \sqrt{\frac{n^2\pi^2 - 6}{3}} \cdot \frac{L}{2n\pi} \quad (14)$$

$$\Delta p = \sqrt{|\langle p^2 \rangle - \langle p \rangle^2|} = \frac{n\pi\hbar}{L} \quad (15)$$

$$\Delta x \Delta p = \sqrt{\frac{n^2\pi^2 - 6}{3}} \cdot (\hbar/2) \geq \frac{\hbar}{2} \quad (\text{for all } n \geq 1). \quad (16)$$

To no one's surprise, (16) verifies the validity of the Heisenberg uncertainty relation for the particle confined in a box.

3 Particle in a box - Solution with Space-Time Lattice

Next we proceed to obtain the solution of (8), with boundary conditions (6) under the assumption of quantized space-time.

We use the word “quantized” as synonymous with discreteized. Thus our quantized x -domain would simply be the discreteized real line given by $R_{\lambda_0} = \{j\lambda_0 \mid j \in \mathbf{Z}\}$, where λ_0 is the lattice parameter introduced earlier. (The lattice parameter λ_0 could be a “fundamental” length, such as the Planck length [10, 11, 9]) and j is an arbitrary integer. For simplicity, we express the quantized time domain in terms of $\tau_0 = \lambda_0/c$, c being the velocity of light. Thus $R_{\tau_0} = \{j_t\tau_0 \mid j_t \in \mathbf{Z}\}$. Note that for any integer j , the quantity $j\lambda_0$ corresponds to a particular point on the real line \mathbf{R} . Similarly $j_t\tau_0$ gives us a specific value for time t for any integer j_t . The continuum limit would obtain if $j \rightarrow \infty$. and $\lambda_0 \rightarrow 0$ then $j\lambda_0 \rightarrow x$, where x is a real number, $0 \leq x \leq L$. Similarly for j_t and $j_t\tau_0$.

Eigenvalues and Eigenfunctions with Space-time Lattice:

The time-dependence of functions $T(t)$ in the space-time setting can be presented as a simple (central¹) difference equation, (we use superscript q to denote results under our space-time lattice model) thus:

$$\frac{T^q((j_t + 1)\tau_0) - T^q((j_t - 1)\tau_0)}{2\tau_0} + i\omega T^q(j_t\tau_0) = 0 \quad (17)$$

or

$$T^q((j_t + 1)\tau_0) - T^q((j_t - 1)\tau_0) + 2i\tau_0\omega T^q(j_t\tau_0) = 0 \quad (18)$$

¹See for example [12]. It turns out the central difference representation for the differential operators leads to a hermitian representation for all relevant physical quantities.

The solution of the above difference equation can be readily seen to be

$$T^q(j_t\tau_0) = T_0^q e^{ij_t\theta}, \quad \theta = \arctan\left(\frac{-w\tau_0}{\sqrt{1-w^2\tau_0^2}}\right) \quad (19)$$

where we have set $T_0^q = T^q(0)$, the value of the function at time zero, in agreement with (7). We note that for small λ_0 (and therefore small τ_0 , the above solution reduces to:

$$T^q(j_t\tau_0) = T_0^q e^{-iwj_t\tau_0} \quad (20)$$

We will not have much occasion to discuss the time dependence of eigenfunctions in this paper. We assume that the length L of the box is an integer multiple of λ_0 . Thus

$$L = J_0\lambda_0 \quad (21)$$

where J_0 is a positive integer. For the quantized real line we can restate the boundary conditions (6) as

$$u^q(j\lambda_0) = \begin{cases} 0 & \text{if } j \leq 0 \\ 0 & \text{if } j \geq J_0, \end{cases} \quad (22)$$

where $j \in \mathbf{Z}$.

Next, we replace the ‘second derivative’ on the left hand side of (8) by the ‘second (central) difference’ with the prescription:

$$\frac{d^2u(x)}{dx^2} \longrightarrow \frac{u^q((j+2)\lambda_0) - 2u^q(j\lambda_0) + u^q((j-2)\lambda_0)}{4\lambda_0^2}. \quad (23)$$

It is significant to point out that we are employing the “central difference”. The various differences are defined in page 7 of Hildebrand [12]. Carrying out the replacement in (8) and multiplying through by $(4\lambda_0^2)$, we have:

$$u^q((j+2)\lambda_0) - 2(1 - 2k^2\lambda_0^2)u^q(j\lambda_0) + u^q((j-2)\lambda_0) = 0 \quad (24)$$

where

$$k^2 = \frac{2mE}{\hbar^2} \quad (25)$$

The above equation is a difference equation of second order with constant coefficients and can be solved by standard methods. The main steps are outlined below. We use the ansatz $u^q(j\lambda_0) = \alpha^j$. Then the equation (24) becomes

$$\alpha^{j+2} - 2(1 - 2k^2\lambda_0^2)\alpha^j + \alpha^{j-2} = 0 \quad (26)$$

from which we infer that

$$\alpha = e^{i\theta/2}, \quad (27)$$

where θ is given by

$$\tan \theta = \pm \frac{2k\lambda_0^2 \sqrt{1 - k^2\lambda_0^2}}{1 - 2k^2\lambda_0^2}. \quad (28)$$

The general solution of (24) can thus be exhibited as:

$$u^q(j\lambda_0) = Ae^{ij\theta/2} + Be^{-ij\theta/2}, \quad (29)$$

where θ is given by (28). Imposing the boundary conditions (21), we get

$$A + B = 0 \quad (30)$$

$$Ae^{iJ_0\theta/2} + Be^{-iJ_0\theta/2} = 0 \quad (31)$$

which lead to the equation:

$$e^{iJ_0\theta} = 1 \quad (32)$$

with the solution

$$\theta = \frac{2n\pi\lambda_0}{J_0}, \quad \text{with } n = 1, 2, 3, \dots \quad (33)$$

The Eigenvalue Equation with Space-time Lattice:

We thus obtain the eigenvalue equation:

$$\frac{2k\lambda_0 \sqrt{1 - k^2\lambda_0^2}}{1 - 2k^2\lambda_0^2} = \tan \frac{2n\pi\lambda_0}{L}. \quad (34)$$

We can solve (34) to obtain a closed expression for k^2 and consequently E_n as shown below. For convenience, we define

$$y = k^2\lambda_0^2 \quad \text{and} \quad t = \tan \left(\frac{2n\pi\lambda_0}{L} \right). \quad (35)$$

The equation (34) then becomes:

$$\frac{2\sqrt{y}\sqrt{1-y}}{1-2y} = t \quad (36)$$

and we obtain

$$y = \frac{1 + t^2 - \sqrt{1 + t^2}}{2(1 + t^2)}. \quad (37)$$

Substituting for y and t in (37), and observing that

$$1 + t^2 = 1 + \tan^2 \left(\frac{2n\pi\lambda_0}{L} \right) = \sec^2 \left(\frac{2n\pi\lambda_0}{L} \right)$$

gives us the following equation.

$$k^2 = \frac{1}{2\lambda_0^2} \left(1 - \cos \left(\frac{2n\pi\lambda_0}{L} \right) \right) \quad (38)$$

$$\Rightarrow E_n^q = \frac{\hbar^2}{4m\lambda_0^2} \left(1 - \cos \left(\frac{2n\pi\lambda_0}{L} \right) \right), \quad n = 1, 2, 3, \dots \quad (39)$$

Thus, as in the continuum case, the variable k is discretized by the requirement that the solution of (24) satisfy the boundary conditions (30, 31). Expanding the right hand side of (39) in powers of λ_0 , we obtain

$$E_n^q = \frac{\hbar^2}{4m} \left(\frac{2n^2\pi^2}{L^2} - \frac{2n^4\pi^4}{3L^4}\lambda_0^2 + O(\lambda_0^4) \right) \quad (40)$$

$$= \frac{\pi^2\hbar^2n^2}{2mL^2} - \frac{\pi^4\hbar^2n^4}{6mL^4}\lambda_0^2 + O(\lambda_0^4), \quad n = 1, 2, 3, \dots \quad (41)$$

The second term in the preceding equation is the leading correction to E_n , due to the assumed fundamental length λ_0 , which turns out to be $O(\lambda_0^2)$. We note that our results for correction to eigenvalues of ‘particle in box’ problem are consistent with the corrections of hydrogen atom spectrum (also $O(\lambda_0^2)$) obtained by Maziashvili et al [5] (treating it as a quantum-gravitational effect) and by Akhoury et al [6] (using minimum-length deformed QM) ².

In the limit as $\lambda_0 \rightarrow 0$, we have $E_n^q \rightarrow E_n$, (see (12)) a result which serves as a check on our calculation.

Since k^2 is nonnegative, $\sqrt{k^2} = \pm k$ is a real number. With k determined by (38), the eigenfunctions $u_n^q(j\lambda_0)$ can be written as

$$u_n^q(j\lambda_0) = A^q \sin(n\pi j\lambda_0/L), \quad n = 1, 2, 3, \dots \quad (42)$$

Normalization and Orthogonality of q -Eigenfunctions :

To determine the normalization factor, A^q , and to check the orthogonality of the eigenfunctions $u_n^q(j\lambda_0)$, we have to compute the following sum. This is somewhat tedious to do by hand but use of

²See [5] for a discussion of sign difference between the results for corrections obtained by [5] and [6]

Mathematica (or some other similar mathematical software) makes the analysis quite simple. We obtain:

$$S_{nn'} = \sum_{j=0}^{J_0} u_n^{q*}(j\lambda_0) u_n^q(j\lambda_0) \quad (43)$$

$$= |A^q|^2 \sum_{j=0}^{J_0} [\sin(n'\pi j\lambda_0/L) \sin(n\pi j\lambda_0/L)] \quad (44)$$

$$= |A^q|^2 \left(\frac{L}{2}\right) \delta_{nn'} \quad (45)$$

where $\delta_{nn'}$ is the usual Kronecker delta symbol. It is then clear that the q -eigenfunctions $u_n^q(j\lambda_0)$ are indeed mutually orthogonal for $n \neq n'$ and that the normalization factor $|A^q|$ is given by

$$|A^q| = \sqrt{\frac{2}{L}}$$

which turns out to be the same as in the continuum case.

4 Computation of Uncertainties with Space-Time Lattice

In the present section we calculate the uncertainties Δx and Δp with q -eigenfunctions in space-time lattice as given in (42). As in the continuum case, we compute

$$\langle p \rangle_n, \langle p^2 \rangle_n, \langle x \rangle_n, \text{ and } \langle x^2 \rangle_n$$

and using these we obtain Δx and Δp as usual.

Computation of Δp :

Recall that the operator p in central derivative representation takes the form

$$p u_n^q(j\lambda) \rightarrow -i\hbar \nabla u_n^q(j\lambda_0) \quad (46)$$

$$\rightarrow -i\hbar \frac{u_n^q((j+1)\lambda_0) - u_n^q((j-1)\lambda_0)}{2\lambda_0} \quad (47)$$

Using (42), we obtain

$$\langle p \rangle_n = \sum_{j=0}^{J_0} u_n^{q*}(j\lambda_0) p u_n^q(j\lambda_0) \quad (48)$$

$$= \frac{|A^q|^2}{2\lambda_0} \left[\sum_{j=1}^{J_0} \sin(n\pi(j-1)\lambda_0/L) \sin(n\pi j\lambda_0/L) \right. \quad (49)$$

$$\left. - \sum_{j=1}^{J_0} \sin(n\pi j\lambda_0/L) \sin(n\pi(j-1)\lambda_0/L) \right] \quad (50)$$

where we added and subtracted some terms to the two sums in (50), which vanish because of boundary conditions (22). Thus we see that $\langle p \rangle_n = 0$.

We can compute $\langle p^2 \rangle_n$ by evaluating the sum directly. We can also get the answer more readily by observing that

$$\langle p^2 \rangle_n = \langle 2mE \rangle_n = \hbar^2 k^2 = \frac{\hbar^2}{2\lambda_0^2} [1 - \cos(\frac{2n\pi\lambda_0}{L})] \quad (51)$$

Thus

$$(\Delta p)^2 = \langle p^2 \rangle_n - \langle p \rangle_n^2 \quad (52)$$

$$= \frac{n^2 \pi^2 \hbar^2}{L^2} - \frac{n^4 \pi^4 \hbar^2}{3L^4} \lambda_0^2 + O(\lambda_0^4) \quad (53)$$

and therefore

$$\Delta p = \left(\frac{\hbar}{\lambda_0} \right) \sin\left(\frac{n\pi\lambda_0}{L}\right) \quad (54)$$

$$= \frac{\pi n \hbar}{L} - \frac{\pi^3 n^3 \hbar}{6L^3} \lambda_0^2 + O(\lambda_0^4) \quad (55)$$

Computation of $\langle x \rangle_n$:

Since

$$u_n^{q*}(j\lambda_0) x u_n^q(j\lambda) \rightarrow u_n^{q*}(j\lambda_0) j \lambda_0 u_n^q(j\lambda_0) \quad (56)$$

using (42), and recalling that

$$a = (n\pi\lambda_0/L). \quad (57)$$

we obtain

$$\langle x \rangle_n = \lambda_0^2 \sum_{j=0}^{J_0} j u_n^{q*}(j\lambda_0) u_n^q(j\lambda_0) \quad (58)$$

$$= |A^q|^2 \lambda_0^2 \sum_{j=0}^{J_0} j \sin^2(a j) \quad (59)$$

The sum $\sum_{j=0}^{J_0} j \sin^2(a j)$ can be evaluated by standard techniques and we obtain:

$$\sum_{j=0}^{J_0} j \sin^2(a j) = \frac{L^2}{4} \quad (60)$$

where we have used the definition $J_0 = \frac{L}{\lambda_0}$ to simplify the final sum. Substituting for $|A^q|^2$ in (59), we get:

$$\langle x \rangle_n = \frac{L}{2} \quad (61)$$

Computation of $\langle x^2 \rangle_n$:

Now since

$$u_n^{q*}(j\lambda_0)x^2u_n^q(j\lambda) \rightarrow u_n^{q*}(j\lambda_0)(j\lambda)^2u_n^q(j\lambda_0), \quad (62)$$

using (42), we obtain

$$\langle x^2 \rangle_n = \lambda_0^3 \sum_{j=0}^{J_0} j^2 u_n^{q*}(j\lambda_0) u_n^q(j\lambda_0). \quad (63)$$

$$= |A^q|^2 \lambda_0^3 \sum_{j=0}^{J_0} j^2 \sin^2(a_j) \quad (64)$$

The sum $\sum_{j=0}^{J_0} j^2 \sin^2(a_j)$ can be evaluated by standard techniques and we obtain:

$$\sum_{j=0}^{J_0} j^2 \sin^2(a_j) = \frac{L^3}{6} + \frac{L\lambda_0^2}{12} - \left(\frac{L\lambda_0^2}{4} \right) \csc^2\left(\frac{\lambda_0 n \pi}{L}\right) \quad (65)$$

$$= \left(\frac{L^3}{6} - \frac{L^3}{4n^2\pi^2} \right) - \left(\frac{n^2\pi^2}{60L} \right) \lambda_0^4 - O(\lambda_0^6) \quad (66)$$

Substituting for $|A^q|^2$, and rearranging, we obtain:

$$\langle x^2 \rangle_n = \frac{L^2}{3} + \frac{\lambda_0^2}{6} - \left(\frac{\lambda_0^2}{2} \right) \csc^2\left(\frac{\lambda_0 n \pi}{L}\right) \quad (67)$$

Using (61) and (67), we get:

$$(\Delta x)^2 = \langle x^2 \rangle_n - (\langle x \rangle_n)^2 \quad (68)$$

$$= \left(\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2} \right) - \left(\frac{n^2\pi^2}{30L^2} \right) \lambda_0^4 + O(\lambda_0^6) \quad (69)$$

which leads to

$$\Delta x = \left(\frac{1}{2\sqrt{3}} \right) \sqrt{L^2 + 2\lambda_0^2 - 6\lambda_0^2 \csc^2\left(\frac{\lambda_0 n \pi}{L}\right)} \quad (70)$$

$$= \left(\sqrt{\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}} \right) - \left(\frac{n^2\pi^2}{60 \left(L^2 \sqrt{\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}} \right)} \right) \lambda_0^4 + O(\lambda_0^6) \quad (71)$$

Using equations (70) and (46), we can write:

$$\Delta x \Delta p = \left(\frac{\hbar}{2\sqrt{3}\lambda_0} \right) \left(\sqrt{L^2 + 2\lambda_0^2 - 6\lambda_0^2 \csc^2\left(\frac{\lambda_0 n \pi}{L}\right)} \right) \sin\left(\frac{n\pi\lambda_0}{L}\right) \quad (72)$$

$$= \left[\left(\sqrt{\frac{n^2\pi^2 - 6}{3}} \right) - \left(\frac{n^2\pi^2 \sqrt{\frac{n^2\pi^2 - 6}{3}}}{6L^2} \right) \lambda_0^2 \right] \left(\frac{\hbar}{2} \right) + O(\lambda_0^4) \quad (73)$$

$$= \sqrt{\frac{n^2\pi^2 - 6}{3}} \left[1 - \left(\frac{n^2\pi^2}{6L^2} \right) \lambda_0^2 \right] \left(\frac{\hbar}{2} \right) + O(\lambda_0^4) \quad (74)$$

The first term in the square bracket in (73) gives us the continuum result while the second term is the correction because of the assumed space-time lattice and is of second order in the lattice parameter λ_0 .

5 Particle in a box: Summary

With our model of quantized space-time as described in the foregoing, we have obtained the q -eigenvalues and q -eigenfunctions for the problem of a particle in a box as a function of the lattice parameter λ_0 . We have shown that the q -eigenfunctions form an ortho-normal set. We have also computed the uncertainties Δx and Δp for the problem. The modification of the Heisenberg uncertainty relation induced by the assumption of space-time lattice for the box problem is shown in (72) and is of second order in lattice parameter λ_0 as presented in (74).

By examining the limit process as $\lambda_0 \rightarrow 0$, we can see that the q -eigenvalues reduce to the ones obtained for the case of continuum domain. Same holds for the q -eigenfunctions as also for the expressions for $\Delta x, \Delta p$. This reflects the continuous nature of the results for eigenfunctions, eigenvalues and uncertainties obtained for the space-time lattice, as a function of the lattice parameter λ_0 .

The first order modification of product $\Delta x \Delta p$ induced by the assumed space-time lattice is of the form

$$\Delta x \Delta p = \alpha \left(1 + \beta \lambda_0^2 \right) \left(\frac{\hbar}{2} \right)$$

where the coefficients α and β for the ‘particle in box’ problem, β can be read off from (74) as

$$\alpha = \sqrt{\frac{n^2\pi^2 - 6}{3}} \quad (75)$$

$$\beta = - \left(\frac{n^2\pi^2}{6L^2} \right) \quad (76)$$

The parameters α and β , of course, depend on the specific problem that we are investigating. However the form of correction of the product of uncertainties above resembles the one given in [1] as well as other papers exploring consequences of minimum length in uncertainty relation (see, for example, [5] and [6]). It is interesting to note that even a simplistic model for space-time lattice and a semi-classical approach to QM seems to reveal shadows of deeper theories.

6 Concluding Remarks

Motivated by the possible existence of a fundamental length, such as Planck length, as a physical reality, we pose the question: how may a fundamental length modify the results in elementary quantum mechanics problems familiar to us? The investigation of the implications of space-time lattice from many different perspectives continues to draw considerable attention in literature (See for example [1] and references cited there). In this paper, we have explored the “particle in a box” problem in some detail with a simple model for space time lattice and obtained corrections induced by the existence of a space-time lattice to the results for eigenvectors eigenvalues, average uncertainties given under the space-time continuum. We have exhibited how the Heisenberg uncertainty relation in space-time continuum compares with the corresponding result in space-time lattice.

For the “particle in a box” problem with space-time lattice, we found that the corrections to eigenvalues etc are of second order in the lattice parameter λ_0 , and although the corrections are observable in principle, the rather small magnitude of Planck length, the lattice parameter $\lambda_0 \approx 10^{-35}$ renders it difficult if not impossible to measure at present. All the same, what one can hope is that such an investigation might give us a pointer, however faint, to a future correct theory which incorporates fundamental length, quantum mechanics as well as theory of relativity. To that end, it appears that it would as well be useful to investigate other elementary quantum mechanics problems ³. For instance, it would be interesting to bring out the λ_0 -dependence of eigenvalue spectrum and eigenfunctions of a simple harmonic oscillator or hydrogen atom in our model and compare them with other approaches to consequences of fundamental length in QM (see for example [5] and [6]). Equally interesting would be to investigate if λ_0 -dependence of any physical quantity is measurable in any experiment in the foreseeable future.

³For a brief discussion of the free particle under the lattice assumption, see [13].

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